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LETTER TO THE EDITOR

Turbulent diffusion as a random-walk process

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Abstract. A random-walk type of diffusion coefficient is shown to reduce to the classical, continuum result for turbulence, in the limit of long diffusion times. The analysis is carried out in a Lagrangian frame, where the mean-square displacement may be calculated explicitly, and the frequency of such displacements obtained from the Rice–Kac theorem. The further problem of finding the random-walk diffusivity in terms of Eulerian variables is considered.

Turbulent diffusion is usually studied in a Lagrangian coordinate system (ie one which follows the motion of the particle), where some exact results may be obtained. In particular, when the velocity is a stationary random function of position and time, Taylor's (1921) well known analysis (in modified form, Kampé de Feriet 1939) yields the variance of particle position, in one dimension, as

$$\langle X^2(t)\rangle = 2\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \langle v(t')v(t'-t'')\rangle = 2\langle v^2\rangle \int_0^t (t-\tau)R_{\mathrm{L}}(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $R_{\mathrm{L}}(\tau)$ is the Lagrangian correlation function. For $t \gg \tau_{\mathrm{L}}$ (where $\tau_{\mathrm{L}} = \int_0^\infty R_{\mathrm{L}}(\tau)\,\mathrm{d}\tau$), equation (1) leads to the important result that the diffusion process is Fickian with constant diffusivity (Hinze 1959)

$$D_{\mathrm{T}} = \langle v^2\rangle \tau_{\mathrm{L}}. \tag{2}$$

As this result is of little use in applications, due to the general problem of relating Lagrangian and Eulerian correlations, the practical response has been to base phenomenological theories on the observed normal distribution of particle displacements from an initial location. However, this type of approach normally fails to predict a quantitative form of diffusion coefficient (Hinze 1959).

Recently, a new approach has been followed by Hutchinson *et al* (1971), who study the deposition of small particles from a turbulent pipe flow. They argue that the diffusion process is a random walk, in the limit of many steps, and apply Chandrasekhar's (1943) form of the diffusion equation. The (one-dimensional) diffusivity is then given by

$$D_{\mathrm{C}} = \frac{n\langle l^2\rangle}{2} \tag{3}$$

where $\langle l^2\rangle$ is the mean-square displacement of the particles and n is the number of displacements per unit time. Both n and $\langle l^2\rangle$ are then arbitrarily expressed in terms of the friction velocity and Townsend's length scale for large eddies.

For the special case of stationary, homogeneous turbulence, one can do rather better. The Lagrangian integral scale, τ_L, is normally interpreted as a 'mean free time'. On this basis we may use Taylor's analysis to calculate $\langle l^2 \rangle$ for $t \gg \tau_L$, thus:

$$\langle l^2 \rangle = \langle (X(t+\tau_L) - X(t))^2 \rangle = 2\langle v^2 \rangle \tau_L^2, \tag{4}$$

and then, from (3), $D_C = n\langle v^2 \rangle \tau_L^2$. Intuitively, we might put $n \sim \tau_L^{-1}$ and recover (2) from (3). However, this seems rather a crude step for what is essentially a continuum process. Instead, we estimate n from the Rice–Kac formula (Rice 1954) for the frequency of the zeros of a random function. We have:

$$n = \frac{1}{\pi}\left[-\frac{\partial^2 R_L(t)}{\partial t^2}\right]_{t=0}^{1/2} = \frac{\sqrt{2}}{\pi}\frac{1}{t_L}, \tag{5}$$

where t_L is the Lagrangian micro-scale and the last step follows by its definition (Hinze 1959).

Substitution of (5) and (4) into (3) shows that the 'random-walk' (D_C) and 'continuous-diffusion' (D_T) forms of the diffusion coefficient are equal at long times, provided $t_L/\tau_L = \sqrt{2}/\pi = 0{\cdot}45$. Measurements of Lagrangian quantities are few and, in general, rather unreliable. Among the best would seem to be those of Shlien and Corrsin (1974), who found $t_L/\tau_L = 0{\cdot}69$.

Our second point concerns the practical application of equation (3). For this form to be useful, we need a way of expressing (3) in terms of Eulerian variables. This means that, like Hutchinson *et al* (1974), we are forced to make assumptions.

First, we would make the suggestion that such assumptions, although unrestricted by the nature of the turbulence, should nevertheless be chosen to yield the correct form for the special case of homogeneous turbulence. In this case, introducing the Lagrangian length scale $\lambda_L = \langle v^2 \rangle^{1/2}\tau_L$, we may make the replacements (Hinze 1959) $\langle v^2 \rangle = \langle u^2 \rangle$, where u is the Eulerian velocity, and $\lambda_E = \beta\lambda_L$, where λ_E is an Eulerian integral length-scale and $\beta \sim 1$. Then, *for homogeneous turbulence only*, equation (2) reduces to

$$D_T = \beta^{-1}\langle u^2 \rangle^{1/2}\lambda_E. \tag{6}$$

In treating equation (3) in an Eulerian frame, we now make assumptions which are slightly different from those of Hutchinson *et al* (1971). By analogy with the Lagrangian analysis we assume $\langle l^2 \rangle = c^2\lambda_E^2$, where c is a constant. On average, the particle will take time $t_E = c\lambda_E\langle u^2 \rangle^{-1/2}$ to travel the distance $c\lambda_E$ and so we put $n = t_E^{-1}$. With these assumptions, we obtain an Eulerian form for the Chandrasekhar diffusion coefficient. That is. from equation (3)

$$D_C = \tfrac{1}{2}c^{-1}\lambda_E^{-1}\langle u^2 \rangle^{1/2}c^2\lambda_E^2 = \tfrac{1}{2}c\langle u^2 \rangle^{1/2}\lambda_E \tag{7}$$

which agrees with equation (6) (the correct form for homogeneous turbulence), if $c = 2\beta^{-1} \sim 2$.

Coincidentally, for the particular case of pipe flow, equation (7) does not differ much from Hutchinson's form of D_C. This is because the RMS radial velocity is fairly constant, with respect to radial position, and roughly equal to the friction velocity. Likewise, the various integral length scales are nearly constant and of much the same magnitude. It would, however, be interesting to make a comparison of the two diffusivities in a more sensitive problem: say, predicting the radial variation of particle number density, when diffusion is from a point source. So far we have found that equation (7) gives very good results for heat or trace-gas diffusion from a point source in a turbulent jet (Davidson and McComb 1974).

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